Volatility models

Working paper

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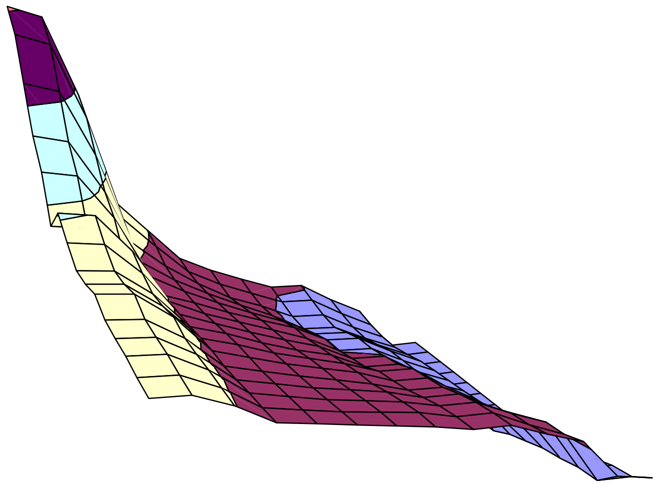

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0 Legend

In the following we use the next notations:

$S(t)$	Equity spot price, financial index…..
$\sigma(t)$	Volatility
$V(t)$	Variance
C	European call option price
K	Strike price.
$W_{1,2}$	Standard Brownian movements.
r	Interest rate.
q	Dividend yield.
κ	Mean reversion rate.
θ	Long run variance
σ_0	Initial volatility
γ	Volatility of volatility (volga)
ρ	Correlation parameter
t_0	Current date.
T	Maturity date.

1 Introducing volatility

The concept of volatility is probably the most investigated phenomenon in modern day finance. The vast interest in volatility is motivated by two related reasons. The growing number of companies utilizing risk management and the huge amount of derivatives traded on today's exchange markets. The fair price of an option is determined by a number of factors including the volatility of the underlying asset. All these factors except the volatility are directly observable from the market or specified by the option contract. The precise account of the volatility is notoriously difficult, but crucial for the correct risk evaluation and options pricing.

2 Constant volatility models

Let us briefly specify some details. The word volatility means rapid or unexpected changes.
At present there exist several notions of volatility.

2.1 Actual volatility (AV)

A measure of randomness or uncertainty or irregularity in the behavior of price at present time. Usually AV is approximated by historical volatility, but this is correct only for the stationary time series. So in practice AV is unobservable.

2.2 Historical volatility (HV)

Volatility, determined on a basis of the past price levels. Two numbers are necessary for calculation of HV. The first - what value of the price we take on the current bar - for example: close, (high+low)/2, (open+high+low+close)/4 and so on. The second number is the range of a time window. It is not clear how long should be the time of sampling. However, it is common to use the past 20 or 60 days period.

Classical estimator: it requires only closing prices C_t and can be defined as the standard deviation σ_c of the daily price returns for a period of time T

$$\begin{aligned} u_t &= \ln(C_t / C_{t-1}) \\ \bar{u} &= \frac{1}{T}\sum_{t=1}^{T} u_t \\ \sigma_c^2 &= \frac{1}{T}\sum_{t=1}^{T}(u_t - \bar{u})^2. \end{aligned} \tag{2.1}$$

This is the optimal (maximum likelihood) estimator obtained from random walk model.

Parkinson (1980): this estimator uses extreme value, the highs H_t and the lows L_t during the day.

$$\sigma_p^2 = \frac{1}{4T\log(2)}\sum_{t=1}^{T}(\log(H_t / L_t))^2 . \tag{2.2}$$

This is five times more efficient than the close-to-close estimate. (That means, for the same amount of data the variance of the data is one fifth that of the close-to-close measure.)

Garman & Klass (1980): this estimator 7.4 times more efficient than close to close

$$\sigma_{gk}^2 = \frac{1}{T}\sum_{t=1}^{T}(0.511(\ln(H_t / L_t))^2 - 0.019\ln(C_t / O_t)\ln(H_t L_t / O_t^2) - 2\ln(H_t / O_t)\ln(L_t / O_t)). \quad (2.3)$$

Rogers & Satchell (1991): Rogers & Satchel estimator is independent of the drift

$$\sigma_{rs}^2 = \frac{1}{T}\sum_{t=1}^{T}(\ln(H_t / C_t)\ln(H_t / O_t) + \ln(L_t / C_t)\ln(L_t / O_t)). \quad (2.4)$$

HV is rather easy to count up and interpret for stationary time series. Unfortunately, the financial time series are not stationary.

2.3 Future volatility (FV)

Future volatility is AV over period to expiry of the option. In the Black-Scholes world some average value of FV defines the option price. FV is a key notion because it is a measure of uncertainty about future price movements, because it is directly related to the risk associated with holding financial securities.
Unknown.

2.4 Implied volatility (IV).

The implied volatility is the attempt to estimate FV on the basis of Black-Shcoles model. It can be made if option is a traded asset. While the BS equation was originally intended to be used in calculating derivative prices from fixed volatility estimates, it can also be used in the opposite way to calculate volatilities from actual derivative prices in the market. If we substitute in the BS formula option market price C_m and other parameters we get the nonlinear equation for volatility σ :

$$C_m = C(S, K, r, \sigma, T). \quad (2.5)$$

Provided that the market value of the option is $C_m \geq C(S, K, r, 0, T)$ there is one value for σ that makes the theoretical option value and the market price the same. The solution is simply enough to find, for example, by bisection method. A problem is that more than a few assumptions incorporated in the Black-Scholes model are not carried out in practice. That shows out some “holes” (using expression of F. Black) in the model. For these are experimentally established fact that:
the distributions of the assets returns have nonzero third and fourth moments;
the value of IV changes with time to maturity;
the value of IV changes with strike (smile effect).
Volatility smile reflects non-lognormal distribution of the returns. Two numbers: mean and volatility parameterise lognormal distribution. The volatility smile is a payment for this simplicity. It is necessary to correct all distribution for everyone strike. Naturally, the second moment of underlying distribution should not change with the strike. And it is so actually in the stochastic volatility models with the nonzero 3rd and 4th moments. In other words IV is the volatility, substituted in the wrong formula to obtain the actual option price asked by financial markets. But IV is the decisive factor for the option trade in spite of the fact that BS model is inaccurate.

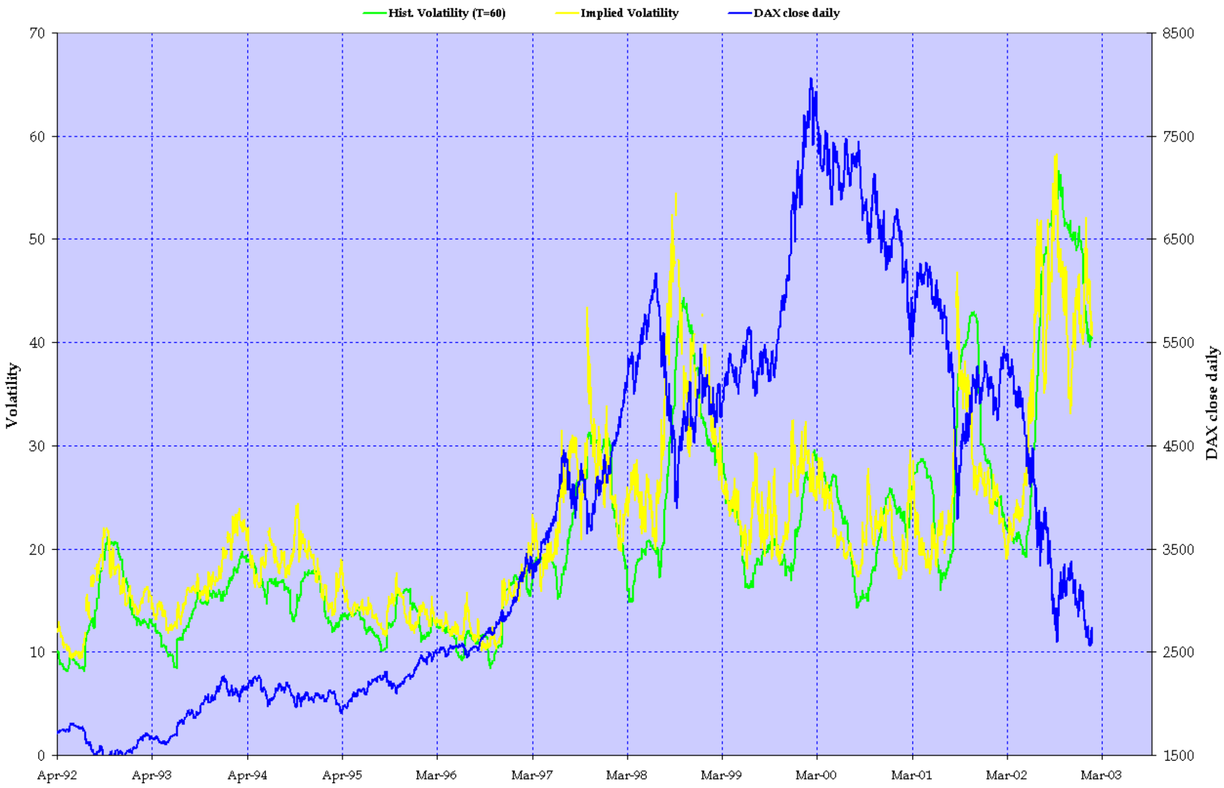

Figure 1: the performance of the DAX index (right scale), annualized 60-day historical volatility based on the DAX close daily data (left scale) and VDAX implied volatility index (left scale) over the representative ten-years period

3 Non-constant volatility models

There is plenty of evidence that returns on equities, currencies and commodities are not normally distributed, they have higher peaks and fatter tails than predicted by a normal distribution. This has been cited as evidence for non-constant volatility.

Before the market crash in 87 the Black-Scholes model provided a good description of the Vanilla market: the strike dependency of the implied volatility was negligible. As the graphs show in today's markets out-of-the-money (otm) put options are more expensive (in volatility terms) than otm calls.

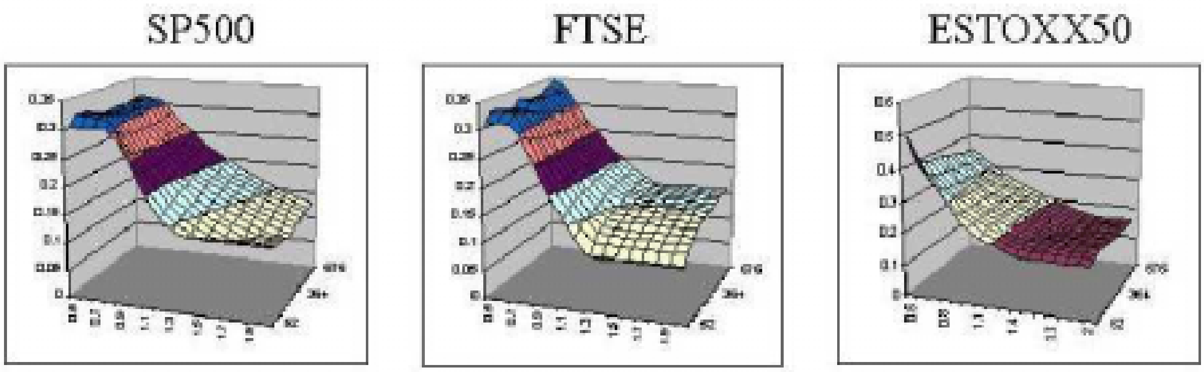

Figure 2: implied volatility surfaces for the key equity indexes

The volatility smile cannot be explained in the framework of normal distribution with constant volatility.

Another reason for introduction of non-constant volatility models is the pricing and hedging of exotic options. Exotic options can be hedged with vanilla options with different strikes and maturities - assuming a continuum of strikes and maturities, and therefore they can only be priced and hedged if the model is consistent with the price of options for all strikes and maturities. Within Black-Scholes we can get the correct price of an option for only one strike and only one maturity. A time-dependent volatility (Merton's model) leads to the correct price of an option for only one strike and all maturities. The need for modelling the volatility in a more general setting is clear, and mainly three distinct types of continuous models have been developed: local volatility models, jump-diffusion models, and stochastic volatility models.

3.1 Local volatility

In the local volatility models the security prices followed a nonlinear diffusion process with drift and volatility depending on the price level and time

$$dS = S(rdt + \sigma(S,t)dW). \tag{3.1}$$

The model is complete but the conjecture of continuous trading is essentially for the local volatility models in view of the fact that the corresponding discrete-time model is incomplete.

For a given maturity T and current stock price S_0, the call option price $C(S_0,K,T)$ is related to the risk-neutral probability density function ("pdf") $p(S_0,S_T,T)$ of the final spot price S_T through the relationship:

$$C(S_0,K,T) = \exp(-rT)\int_K^\infty p(S_0,S_T,T)(S_T - K)dS_T . \tag{3.2}$$

By taking twice the derivative with respect to K, we obtain the Breeden – Litzenberger formula for calculating $p(S_0,K,T)$:

$$p(K,T;S_0) = \exp(rT)\frac{\partial^2 C}{\partial K^2}. \tag{3.3}$$

In the Black-Scholes framework, (3.3) can be calculated analytically and leads to the lognormal density. In general, based on call option prices, it is possible to derive an implied density by using (3.3) directly.

The pdf $p(S_0,S_T,T)$ evolves according to the Fokker-Planck equation:

$$\frac{\partial p}{\partial T} = -\frac{\partial(rS_T p)}{\partial S_T} + \frac{1}{2}\frac{\partial^2(\sigma^2(S_T,T)S_T{}^2 p)}{\partial S_T{}^2}. \tag{3.4}$$

Application of Itô's lemma to the call option price $C(S_0,K,T)$ and the self-financing assumption gives rise to a partial differential equation (PDE) for $C(S_0,K,T)$ which is a straightforward generalization of the famous Black-Scholes PDE

$$\frac{\partial C}{\partial T} = r\left(-K\frac{\partial C}{\partial K}\right) + \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 C}{\partial K^2}. \tag{3.5}$$

Rearranging this we find that:

$$\sigma^2(S,t) = 2\left.\frac{\dfrac{\partial C}{\partial T} + rK\dfrac{\partial C}{\partial K}}{K^2\dfrac{\partial^2 C}{\partial K^2}}\right|_{\substack{K=S\\T=t}} . \qquad (3.6)$$

As the right-hand side of (3.6) can be computed if we assume a continuum of call option prices, this implies that under this assumption, the local volatility is given uniquely by the analytical form (3.6)). This formula was derived by Dupire.

We can view (3.6) as a definition of the local volatility function regardless of what kind of process (stochastic volatility for example) actually governs the evolution of volatility. But if volatility is a deterministic function of stock price and time, then the model is complete: there is only one source of risk, the stock price itself, which can be dynamically hedged the same way as in the Black-Scholes model. Market completeness and uniqueness of local volatility surface are the two crucial implications of this model.

This calculation of the volatility surface from option prices worked because of the particular form of the payoff, the call payoff, which allowed us to derive the very simple relationship between derivatives of the option price and the transition probability density function.

In practice, however, there are only a finite number of liquid European call options in the market, and determining the local volatility surface can be regarded as a function approximation problem from a finite data set with a nonlinear observation function. This is a well-known ill-posed problem: there are typically an infinite number of solutions to the problem. So local volatility models are a theoretical dream and a numerical nightmare.

3.2 Constant elasticity of variance model

As we see above price volatility is non-constant. The second observed feature of stock volatility is correlation between volatility and price levels. Loosely speaking market expects the volatility to rise as the price falls. This is so called leverage effect. In this way leverage introduce negative correlation between price and volatility. This argument provides introduction of constant elasticity of variance (CEV) model.

$$dS = Srdt + \sigma S^{\alpha-1} dW , \qquad (3.7)$$

which is a particular case of local volatility model (3.1), CEV model has a several desirable feature. Firstly it catches leverage effect, secondly as in the all local volatility models the market is complete this means that non-arbitrage argument alone enough to define unique option price, thirdly equation (3.7) analytically tractable.

3.3 Stochastic volatility

Why are more complex volatility models required? The answer is simple – because the prices from stochastic engines are not supported by market prices. As a result financial engineers have to recalibrated model parameters every day to the new market data. It is not consistent with an accurate description of the dynamics. The next (but not the last) step in the volatility models history is the models with stochastic volatility.

Numerous pricing models that treat the volatility as a stochastic variable have been proposed over the past decade. The introduction of a second stochastic process presents new complexity, it is related to the risk-neutral valuation concept. Because we have two sources of randomness (price and volatility) and the volatility is not a traded asset we must hedge our option with two other contracts, one being the underlying asset as usual, but now we also need another option to hedge the volatility risk.

3.4 Heston's stochastic volatility model

In this section we specify Heston's stochastic volatility model

$$dS(t)/S(t) = \mu dt + \sigma(t) dW_1, \quad (3.8)$$

$$d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t) dW_2. \quad (3.9)$$

To take into account leverage effect, Wiener stochastic processes W_1, W_2 should be correlated $dW_1 \cdot dW_2 = \rho dt$. The stochastic model (3.9) for the volatility is related to the square-root process of Feller (1951) and Cox, Ingersoll and Ross (1985). For the square-root process (3.9) the volatility is always positive, and if $2\kappa\theta > \gamma^2$ then it cannot reach zero. Note that, the deterministic part of process (3.9) is asymptotically stable if $\kappa > 0$. Obviously, that equilibrium point is $\sigma^* = \sqrt{\theta}$.

The attractive features of the Heston stochastic volatility model are:

- its volatility updating structure permits analytical solutions to be generated for European options
- the form of the Heston stochastic process used to model price dynamics allows for non-lognormal probability distributions
- Heston stochastic model takes into account the leverage effect
- this model describes important mean-reverting property of volatility
- the empirically observed Black-Scholes volatility surfaces are often looking similar to the ones generated by the Heston model

Clearly, that the Heston's model is a real player in the competition to be a successor of the Black and Scholes model. This model is very popular among practitioners now.

On the other hand there remain some disadvantages and open questions:

- for certain parameter constellations we observed negative option prices or at least prices which were lying below the usual arbitrage bounds (which makes Black-Scholes volatility inversion impossible !)
- the model did not consistently perform well across the various maturity by no means did it eliminate all biases
- Heston's model implicitly takes systematic volatility risk into account by means of a linear specification for the volatility risk premium.

It is worth to note that parameters of Heston stochastic volatility model ($\kappa, \theta, \gamma, \sigma_0$) after calibration to market data turn out to be non-constant. This means that at best we can deduce from the prices of derivatives, so called fitting. But this is far from adequate, the fitting will only work if those who set the prices of derivatives are using the same model and they are consistent in that the fitted parameters do not change when the model is refitted a few days later. Whether we have a deterministic volatility surface or a stochastic volatility model with prescribed or fitted parameters, we will always be faced with how to interpret refitting. Was the market wrong before but is now right, or was the market correct initially and now there are arbitrage opportunities?

4 Conclusions

In the paper we concentrate on continuous-time volatility models. These models offer the natural framework for options pricing. Continuous-time volatility models provide attractive and intuitive description for the observed volatility patterns and observed biases in implied volatility. In particular smiles, skews and implied volatility term structure arise naturally from stochastic volatility models. However stochastic volatility model is non-stationary and therefore can be considered as complex nonlinear function, that “describe” local behavior of the market.